Dr. Shiva Kant Mishra
Deptt. of Physics
H. D. Jain College, Ara

B. Sc. Part III

Paper —

Classical Mechanics

Hamilton's Principle

According to Hamilton's Principle: "The path actually traversed by a conservative, holonomic dynamical system from time t_1 and t_2 is one over which the integral of the Lagrangian between limits t_1 and t_2 is stationary, i.e. the time integral of the Lagrangian is extremum.

Analytically it can be represented as

$$\int_{t_1}^{t_2} L\,dt = J = \text{extremum} \quad \text{--- (1)}$$

where J is the extremum value of the time integral of the Lagrangian and is known as Hamilton Principle function for the path.

Equ^n (1) may be represented as

$$\delta\int_{t_1}^{t_2} L\,dt = 0 \quad \text{--- (2)}$$

where δ is the variation symbol.

This principle helps us to distinguish the actual path from the neighbouring paths.

Deduction of Hamilton's Principle:

Let us consider that the conservative, holonomic dynamical system moves from P to Q, where P and Q are initial and final configurations of the system at times t_1 and t_2 respectively. Let PRQ be the actual path and PR'Q, PR''Q the two neighbouring paths out of infinite number of possibilities.

For the deduction of Hamilton's Principle the following two conditions must be satisfied

1) δt must be equal to zero at end points i.e. at t_1 the particle must be at P and at t_2 the particle must be at Q.

2) δr must be equal to zero at end points, i.e; the points P and Q are fixed in space.

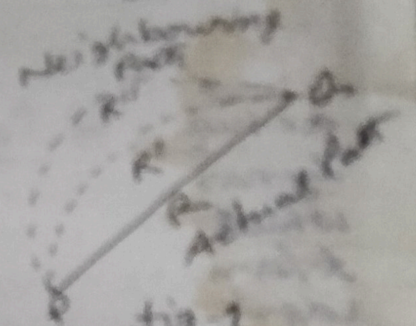

fig 1

Let the system be acted upon by a no. of forces represented by F. Let ith particle of the system acted upon by force F_i acquire acceleration $\ddot{r}_i$, so that we have

$$F_i = m_i \ddot{r}_i$$

From D'Alembert Principle, we have

$$\sum_i (F_i - m_i \ddot{r}_i)\cdot \delta r_i = 0 \quad \text{—— (1)}$$

or $\sum F_i \cdot \delta r_i - \sum m_i \ddot{r}_i \cdot \delta r_i = 0$ —— (1)

but $\ddot{r}_i \cdot \delta r_i = \frac{d}{dt}(\ddot{r}_i \cdot \delta r_i) - \dot{r}_i \cdot \frac{d}{dt}(\delta r_i)$ —— (2)

As there is a little variation along the actual and neighbouring paths, we have

$$\delta r_i = r_i' - r_i \text{ (say).}$$

Then $\frac{d}{dt}(\delta r_i) = \frac{d}{dt}(r_i' - r_i) = \frac{dr_i'}{dt} - \frac{dr_i}{dt}$

$$= \delta\left(\frac{dr_i}{dt}\right) = \delta(\dot{r}_i) \quad \text{—— (3)}$$

Here primes have been used for neighbouring paths. Using (3), (2) may written as

$$\ddot{r}_i \cdot \delta r_i = \frac{d}{dt}(\dot{r}_i \cdot \delta r_i) - \dot{r}_i \cdot \delta(\dot{r}_i) \quad \text{—— (4)}$$

Using above eqn (1) becomes

$$\sum_i F_i \cdot \delta r_i - \sum_i m_i\left(\frac{d}{dt}(\dot{r}_i \cdot \delta r_i) - \dot{r}_i \cdot \delta \dot{r}_i\right) = 0$$

or $\sum_i F_i \cdot \delta r_i - \sum_i m_i\left[\frac{d}{dt}(\dot{r}_i \cdot \delta r_i) - \frac{1}{2}\delta(\dot{r}_i)^2\right] = 0$

or $\sum_i F_i \cdot \delta r_i - \sum_i \frac{1}{2} m_i \delta(\dot{r}_i^2) = \sum \frac{d}{dt}(m_i \dot{r}_i \cdot \delta r_i)$

or $\sum_i F_i \cdot \delta r_i + \delta\left(\sum_i \frac{1}{2} m_i \dot{r}_i^2\right) = \sum_i \frac{d}{dt}(m_i \dot{r}_i \cdot \delta r_i)$ —— (5)

But $\sum F_i \cdot \delta r_i$ = work done by the forces F_i during displacements $\delta r_i = \delta W$ (say) Page No. 2

Also $\sum \frac{1}{2} m_i \dot{r}_i^2$ = Kinetic energy of the system = T.

Therefore eqn (5) becomes

$$\delta W + \delta T = \sum_i \frac{d}{dt}(m_i \dot{r}_i \cdot \delta r_i)$$

Integrating above expression between the limits t_1 and t_2, we get

$$\int_{t_1}^{t_2} (\delta W + \delta T)\, dt = \int_{t_1}^{t_2} \sum_i \frac{d}{dt}(m_i \dot{r}_i \cdot \delta r_i)\, dt$$

$$= \sum \int_{t_1}^{t_2} d(m_i \dot{r}_i \cdot \delta r_i)$$

$$= \sum_i \left[m_i \dot{r}_i \cdot \delta r_i\right]_P^Q = 0$$ since $\delta r_i = 0$

at the end points P and Q.

For a conservative system, we know

$\delta W = -\delta V$ where V is the potential energy.

$$\therefore \int_{t_1}^{t_2} (-\delta V + \delta T)\, dt = 0$$

or $\int_{t_1}^{t_2} \delta(T - V)\, dt$

i.e. $\delta \int_{t_1}^{t_2} (T - V)\, dt = 0$

or $\delta \int_{t_1}^{t_2} L\, dt = 0$

or $\int_{t_1}^{t_2} L\, dt = J =$ extremum

which is Hamilton's Principle.